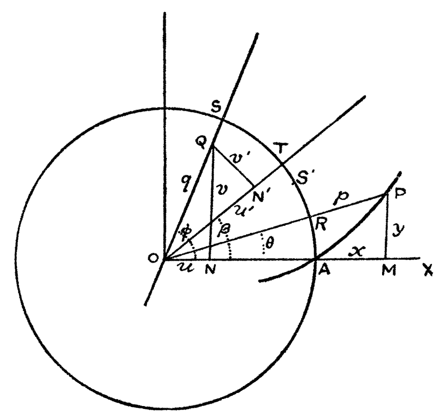
*A Classification of Logarithmic Systems.**

By Irving Stringham.

Though the graphical representation of logarithms by means of the logarithmic spiral is well known, I am not aware that any attempt has been made to use this curve, properly defined as a geometrical locus, as the means for defining the logarithm and demonstrating its properties. The problem turns out to have not only its geometrical interest, but also some importance for analysis in general, giving rise to what I venture to call *gonic* systems of logarithms, whose moduli contain an angular determining-element, and leading, through their introduction, to a classification of logarithmic systems.

Construction and Definitions: The Gonic Systems.—In a circle whose radius is unity, the diameters through S and T are fixed, OR turns about O with a constant velocity, Q moves along OS with a constant velocity, P along OR with a velocity proportional to its distance from O and is supposed to cross the circumference of the circle at A at the instant when Q passes O. Let the velocities of

$$\begin{array}{ll} P \text{ in } OR \text{ at } A & = \lambda, \\ Q \text{ in } OS & = \mu, \\ R \text{ in } ARS & = \omega. \end{array}$$



Let

$$\begin{array}{lll} OM = x, & MP = y, & OP = p, \\ ON = u, & NQ = v, & OQ = q, \\ ON' = u', & N'Q = v', & \text{arc } AS = \phi, \\ \mu/\sqrt{\lambda^2+\mu^2} & = m, & \text{arc } AT = \beta, \\ \overline{OT} = \cos\beta + i\sin\beta = \text{cis}\,\beta = \varepsilon. \end{array}$$

OX is taken as the real axis and

$$z = x + iy, \quad w = u + iv, \quad i = \sqrt{-1}.$$

The condition $\tan(\phi - \beta) = \omega/\lambda$ is assigned arbitrarily; and this, together with $\mu/\sqrt{\lambda^2 + \omega^2} = m$, by elimination of ω, gives also

$$m = \mu/\lambda . \cos(\phi - \beta).$$

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Thus m—as is seen from its value in terms of λ, μ and ω—is the ratio of the velocity of Q in OS to that of P in AP at A, and—as its value $\mu/\lambda . \cos(\phi - \beta)$ shows—it is also the ratio of the velocity of N' along OT to that of P along OR when R is at A.

Since the rates of variation of u' and p, so long as m remains constant, maintain their ratio to each other undisturbed, namely, the ratio m/p, the fixing of m fixes their relative values. Hence m, and β, which is independent of m, are assumed to be constant for all corresponding values of z, w, in any one system of logarithms, whatever the value of ϕ, and fix the system, while λ, μ, ω may vary subject only to the conditions

$$\mu/\sqrt{\lambda^2 + \omega^2} = \text{constant}, \ \tan(\phi - \beta) = \omega/\lambda.$$

The terms modulus, logarithm, base, and exponential are defined as follows:

1. $m\varepsilon$ is the modulus of the system of logarithms.

2. The difference $\overline{OQ} - \overline{OQ'}$ is the logarithm, with respect to modulus $m\varepsilon$, of the ratio $\overline{OP}/\overline{OP'}$, wherein P, P' correspond to Q, Q'. In particular $\overline{OQ}$ is the logarithm, to modulus $m\varepsilon$, of $\overline{OP}$. The symbolic definitions are:

$$w - w' \equiv \log^{(m\epsilon)}(z/z'),$$
$$w \equiv \log^{(m\epsilon)} z.\text{*}$$

By assuming that P passes A at the instant when Q passes the origin, so that to $u = v = 0$ corresponds $y = 0$, $x = 1$, we take advantage of the convenient relation

$$\log^{(m\epsilon)} 1 = 0.$$

3. In any system of logarithms the value of z, for which $w = 1$, is the base of the system; it is here denoted by b_ε.

4. The value of p, for which $u' = 1$, will remain unchanged and independent of β so long merely as m is constant.

Denoting this value of p by b, we define z symbolically as a function of w by the identity

$$z \equiv b_\epsilon^w.$$

The expression b_ϵ^w is here called the exponential of w with respect to base b_ϵ.

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*I have elsewhere written this in the form ${}^{m\varepsilon}\log z$, but this notation is sometimes used to denote logarithm with respect to *base* $m\varepsilon$ (see Harnack, Diff. u. Int. Rechnung, p. 13). It seems best therefore to abandon it in favor of a notation that has not been already used by reputable authors for another purpose.

The Law of Metathesis.—The value of m being unchanged, suppose μ changed to $k\mu$ (k a real quantity), then $\mu/\sqrt{\lambda^2+\omega^2}.\varepsilon$ becomes $k\mu/\sqrt{\lambda^2+\omega^2}.\varepsilon = km\varepsilon$, and q, w become kq, kw, whence

$$\begin{aligned} kw &= k \log^{(m\epsilon)} z \\ &= \log^{(km\epsilon)} z, \end{aligned}$$

and $\therefore$ also

$$= km \log^{(\epsilon)} z.$$

More particularly

$$w = \log^{(m\epsilon)} z = m \log^{(\varepsilon)} z,$$

from which, by putting $w = 1$, and correspondingly $z = b_\epsilon$, we obtain m in the form

$$m = 1/\log^{(\varepsilon)} b_\epsilon,$$

and the logarithmic function in the consequent form

$$w = \log^{(m\epsilon)} z = \frac{\log^{(\epsilon)} z}{\log^{(\epsilon)} b_\epsilon}.$$

When now for z in this last equation we substitute its exponential equivalent b_ϵ^w, the formula thus obtained,

$$w \log^{(\epsilon)} b_\epsilon = \log^{(\epsilon)} b_\epsilon^w,$$

expresses the law of metathesis, if I may so call it, for the natural system of logarithms with complex modulus. For logarithms in general the similar formula

$$w \log^{(k\epsilon)} b_\epsilon = \log^{(k\epsilon)} b_\epsilon^w$$

is obtained from the preceding by multiplying each member by the real quantity k, the complex quantity b_ϵ representing, in the new system with the new modulus $k\varepsilon$, any complex value whatever.

The Law of Involution.—Successive applications of the law of metathesis give the following equivalent expressions:

$$\begin{aligned} w \log^{(k\epsilon)} b_\epsilon^{w'} &= w' \log^{(k\epsilon)} b_\epsilon^w \\ &= \log^{(k\epsilon)} (b_\epsilon^{w'})^w = \log^{(k\epsilon)} (b_\epsilon^w)^{w'}. \end{aligned}$$

A sufficient condition for the equivalence of the last two is

$$(b_\epsilon^{w'})^w = (b_\epsilon^w)^{w'},$$

and for every pair of values w, w', there exists the corresponding equation as above written, expressing the law of involution. We may therefore omit the parentheses and write without ambiguity

$$(b_\epsilon^{w'})^w = b_\epsilon^{w'w} = b_\epsilon^{ww'}.$$

The Addition Theorem.—The definition of the logarithmic function states also in substance the addition theorem; for the equation of definition is

$$\begin{aligned} w - w' &= \log^{(m\epsilon)} z - \log^{(m\ )} z' \\ &= \log^{(m\epsilon)} (z/z'), \end{aligned}$$

which is the form of the theorem for $w - w'$. In order to obtain the form for $w + w'$, we write in the last equation $w = 0$, whence

$$- w' = \log^{(m\epsilon)} (1/z');$$

but

$$\begin{aligned} + w' &= \log^{(m\epsilon)} z', \\ \therefore\ w + w' &= \log^{(m\epsilon)} z + \log^{(m\epsilon)} z' \\ &= \log^{(m\epsilon)} (z . z'), \end{aligned}$$

which is the form required.

The Index Law follows as a corollary from the addition theorem. Thus if

$$z = b_\epsilon^w \quad , \quad z' = b_\epsilon^{w'},$$

$$\therefore\ w = \log^{(m\epsilon)} z,\ w' = \log^{(m\epsilon)} z'$$

and

$$z . z' = b_\epsilon^w . b_\epsilon^{w'}.$$

But

$$\begin{aligned} \log^{(m\epsilon)} (z . z') &= \log^{(m\epsilon)} z + \log^{(m\epsilon)} z' \\ &= w + w', \\ \therefore\ z . z' &= b_\epsilon^{w + w'} \\ &= b_\epsilon^w . b_\epsilon^{w'}, \end{aligned}$$

which is the index law for multiplication; and the law for division is deduced by a repetition of this process with the signs / and —.

The Agonic System: $\beta = 0$.—The special value zero for the modular angle AOT eliminates the imaginary term from the modulus and introduces the ordinary system of logarithms, whose equations of definitions are

$$w' = \log^{(m1)} z,\ z = b_1^{w'},$$

or as they may now be written without ambiguity,

$$w' = \log^{(m)} z,\ z = b^{w'}.$$

The original systems above described, whose modulus involves an independent angular element, will be here referred to as *gonic* systems; when specialized by the omission of this angular element it will be called the ordinary or *a-gonic* system.

The laws of operation in the a-gonic system have their expression in the formulae already deduced for the gonic systems, with β and ε everywhere dropped out. The geometrical representation is obtained by turning the figure $TN'OQS$ backward through the angle AOT, so that T, S fall into the positions A, S'. The new ϕ, say ϕ', $=$ arc $TS =$ arc AS', the new modulus

$$= \mu / \sqrt{\lambda^2 + \omega^2} = \mu/\lambda . \cos\phi' = m,$$

its former value with the factor ε omitted, no change in λ, μ, and ω having taken place. Thus the motion of P remains intact and the new Q moves in the line OS' with its former velocity. Hence the values of z in the two systems are identical, while w' in the agonic system and w in the gonic bear the relation to each other,

$$w = w' \operatorname{cis} \beta = w'\varepsilon,$$

and we may write

$$w = \log^{(m\varepsilon)} z = \varepsilon \log^{(m)} z.$$

Since ε is independent of m, this equation, together with

$$k \log^{(m\varepsilon)} z = \log^{(km\varepsilon)} z$$

previously demonstrated, expresses the law: *To multiply the modulus of a logarithm by any quantity has the effect of multiplying the logarithm itself by the same quantity.*

The corresponding equation in terms of the exponential functions, namely,

$$b_\varepsilon^w = b^{w/\varepsilon},$$

gives us (by the law of involution) the value of the gonic base b_ε in the form

$$b_\varepsilon = b^{1/\varepsilon}.$$

The value of b itself, expressed in terms of m and e, the base corresponding to modulus 1, is obtained in like manner from the inverse of the equation $1/m = \log^{(\varepsilon)} b_\varepsilon$, namely, from

$$b_\varepsilon = e_\varepsilon^{1/m} = e^{1/m\varepsilon},$$

giving

$$b = e^{1/m}.$$

A second agonic system is obtained by putting $\beta = \pi$. Its modulus is $-m$, its base $1/b$, and its logarithms are the negatives of those in the systems whose modulus and base are $+m$ and b. In every other respect the two systems are alike.

Linear Systems.—When the angle $TOS = 0$, the corresponding values of function and variable are $w = q \operatorname{cis} \beta = q\varepsilon$, and $z = p$ a real quantity; and the relation between p and q is expressed in either of the forms,

$$w = q\varepsilon = \log^{(m\varepsilon)} p, \quad z = p = b_{\varepsilon}^{q\varepsilon},$$

belonging, when β is not zero, to a gonic system. Q moves in OT with the velocity μ, P in OA with the velocity λp. Hence in a gonic system the logarithm of a real quantity is, in general, complex; though in particular, when $\beta = \pm \pi/2$, it is a pure imaginary,

When $\beta = 0$, the condition $\phi = \beta$ makes angle $AOT =$ angle $TOS = 0$, $w = q$, $z = p$, $\omega = 0$, $m = \mu/\lambda$, $\varepsilon = 1$; and the relations between p and q are

$$q = \log^{(m)} p, \quad p = b^q.$$

These may be regarded as belonging to the general agonic system, obtained from it by making $\phi = 0$, or as the defining equations of the ordinary system for real quantities only, whose geometrical representatives are lengths upon the real axis laid off in accordance with the logarithmic law: that q varies uniformly while the rate of change of p bears a constant ratio to its own length. The modulus, μ/λ, is the ratio of the rate of q to that of p at the instant when $p = 1$, and the base, b, is the value of p when $q = 1$. This is the well-known Naperian representation of the logarithmic function.

The Semi-cosine Equivalent of b_{ε}^{w}.—Returning to the general case: ϕ and β unequal and not zero, and denoting the arc AR by θ, $\overline{OP}$ is

$$z = p \operatorname{cis} \theta = b_{\varepsilon}^{w}.$$

Then since the velocity of N' in OT is $\mu \cos(\phi - \beta)$ and that of P in OR is λp, while $m = \mu/\lambda . \cos(\phi - \beta)$, the relation between ON', $= u'$ and OP, $= p$, two real quantities, is that of an exponential to its logarithm, with respect to the modulus m; that is,

$$u' = \log^{(m)} p \text{ and } p = b^{u'}.$$

And again, since $\omega = \lambda \tan(\phi - \beta)$ and $m = \mu/\lambda . \cos(\phi - \beta)$,

$$\therefore\ m\omega = \mu \sin(\phi - \beta);$$

but ω and μ are the rates of change of θ and q respectively, and $q = 0$, $\theta = 0$ are simultaneous values;

$$\therefore\ m\theta = q \sin(\phi - \beta) = v'$$

and
$$\theta = \frac{v'}{m}.$$

Hence
$$b_\epsilon^w = p \operatorname{cis} \theta$$
$$= b^{u'} \operatorname{cis} \frac{v'}{m};$$

or also, introducing $m\varepsilon$ and b_ϵ in place of m and b, and substituting for w its equivalent $\varepsilon u' + i\varepsilon v'$,
$$b_\epsilon^{\varepsilon u' + i\varepsilon v'} = b_\epsilon^{\varepsilon u'} \operatorname{cis} \frac{\varepsilon v'}{m\varepsilon};$$

or again, replacing u', v' by their values, $u \cos\beta + v \sin\beta$, $v \cos\beta - u \sin\beta$, in terms of u, v, β,
$$b_\epsilon^{u+iv} = b^{u\cos\beta + v\sin\beta} \operatorname{cis} \frac{v\cos\beta - u\sin\beta}{m}.$$

When $\beta = 0$, this assumes the familiar form
$$b^{u+iv} = b^u \left(\cos\frac{v}{m} + i \sin\frac{v}{m}\right).$$

The Derivatives of the Logarithmic and Exponential Functions.—By definition, the rates of change of p, q and θ are respectively
$$\frac{dp}{dt} = \lambda p, \quad \frac{dq}{dt} = \mu, \quad \frac{d\theta}{dt} = \omega,$$

in which t is taken to represent time. Differentiating z and w, expressed in terms of p, q, θ and ϕ, we have
$$dz = d\,(p \operatorname{cis} \theta),$$
$$= (dp + ip\,d\theta) \operatorname{cis} \theta$$
$$= (\lambda p + ip\omega) \operatorname{cis} \theta . dt,$$

and
$$dw = d\,(q \operatorname{cis} \phi)$$
$$= dq . \operatorname{cis} \phi$$
$$= \mu \operatorname{cis} \phi . dt,$$
$$\therefore \frac{dw}{dz} = \frac{\mu \operatorname{cis} \phi}{p\,(\lambda + i\omega) \operatorname{cis} \theta},$$

and since $\omega = \lambda \tan(\phi - \beta)$,
$$\therefore \frac{dw}{dz} = \frac{\mu \operatorname{cis} \phi \cos(\phi - \beta)}{p\lambda \operatorname{cis}(\phi - \beta) \operatorname{cis} \theta}$$
$$= \frac{\mu \cos(\phi - \beta) \operatorname{cis} \beta}{\lambda z}$$
$$= \frac{m\varepsilon}{z},$$

the formula for the derivative of the logarithmic function. Hence also, for the exponential function,

$$\begin{aligned}\frac{dz}{dw} &= \frac{z}{m\varepsilon} \\ &= b_\epsilon^w . \log^{(1)} b_\epsilon ;\end{aligned}$$

for it has been shown that $1/m = \log^{(\epsilon)} b_\epsilon = \varepsilon \log^{(1)} b_\epsilon$.

The Locus of P.—The rates of change of p and θ are respectively λp and ω, $= \lambda \tan(\phi - \beta)$; hence by the definition of a logarithm, in real quantities, θ is the logarithm of p with respect to the modulus $\{\lambda \tan(\phi - \beta)\}/\lambda = \tan(\phi - \beta)$; or, in equivalent terms,

$$\theta = \tan(\phi - \beta) . \log^{(1)} p ,$$

which is the equation of the logarithmic spiral. Because $pd\theta/dp = \tan(\phi - \beta)$, this curve crosses its radius vector at an angle $= TOS$.

The Sine and Cosine Functions.—An obvious generalization of the formulae of the circular and hyperbolic functions is obtained by assuming, as definitions of the generalized sine and cosine, the functions

$$\frac{\varepsilon}{2}(b_\epsilon^w - b_\epsilon^{-w}), \quad \frac{1}{2}(b_\epsilon^w + b_\epsilon^{-w}).$$

Let these be denoted for the moment by $\mathrm{sinb}_\epsilon w$ and $\mathrm{cosb}_\epsilon w$ respectively. Well known substitutions and reductions then lead to the following formulae:

$$\begin{aligned} b_\epsilon^w &= \mathrm{cosb}_\epsilon w + \varepsilon^{-1} \mathrm{sinb}_\epsilon w , \\ b_\epsilon^{-w} &= \mathrm{cosb}_\epsilon w - \varepsilon^{-1} \mathrm{sinb}_\epsilon w , \\ \mathrm{cosb}_\epsilon^2 w &- \varepsilon^{-2} \mathrm{sinb}_\epsilon^2 w = 1 , \\ \mathrm{sinb}_\epsilon (w \pm w') &= \mathrm{sinb}_\epsilon w \, \mathrm{cosb}_\epsilon w' \pm \mathrm{cosb}_\epsilon w \, \mathrm{sinb}_\epsilon w' , \\ \mathrm{cosb}_\epsilon (w \pm w') &= \mathrm{cosb}_\epsilon w \, \mathrm{cosb}_\epsilon w' \pm \varepsilon^{-2} \mathrm{sinb}_\epsilon w \, \mathrm{cosb}_\epsilon w' ,\end{aligned}$$

from which others are easily obtained. These include the corresponding formulae in hyberbolic and circular functions as special cases, $b = e$ and $\varepsilon = 1$ giving the hyperbolic, $b = e$ and $\varepsilon = i = \sqrt{-1}$ the circular forms.

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